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Numerical Solution of Systems of Random Differential Equations with Gaussian Statistics

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A direct approach is used to compute a numerical solution for a system of (nonlinear) ordinary differential equations with Gaussian statistics. It is shown that for a large class of problems the accuracy of the solution can be made as great as desired by taking the step size h sufficiently small. © 1985 Academic Press, Inc.

1. INTRODUCTION

Random ordinary differential equations arise in electronics, control theory, chemical engineering, structural analysis, econometrics, life sciences, and other physical and operational processes whenever uncertainties are present in initial conditions or parameters. Without the uncertainties, such systems are treated by the usual numerical methods [1, 2].

The Monte Carlo technique [3] can be used to numerically solve any stochastic o.d.e., but a large number of runs is necessary to achieve a satisfactory confidence level. Variance reduction techniques [4] may be used to increase the probabilistic accuracy of the mean trajectory, but not of the covariance matrix (necessary to describe the assumed Gaussian distribution). The method of covariance analysis describing functions [5, 6] is the best in the sense that the least-squares error of piecewise linear approximation is minimized. Barry and Boyce [7] have a numerical method for certain differential equations with random boundaries.

There is a lot of literature on theoretical questions and analytic approaches, e.g., Adomian [8, 9], Tsokos and Padgett [10], and Bharucha–Reid [11]. Existence and uniqueness of solutions to various problems which model stochastic differential systems are given, but analytical solutions are difficult to obtain. The modelling of the stochastic integral of white Gaussian noise by the Wiener process is one example, and a clear description of this can be found in [12, pp. 146–167]. Perturbation techniques are sometimes used (see [13] for a readable account).

Our method is related to that of perturbation and to the classical

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technique of propagation of error through a system (we propagate it stepwise). In Section 2 we derive the computational procedure via a representation of the system as a first-order state space. In Section 3 we consider the mean square error of approximation. We present examples in Section 4 along with some computational aspects.

2. Derivation of the Method

The method can be applied to a system of any number of linear or nonlinear random differential equations of various mixed orders, but we demonstrate the derivation here in the case of the following system:

$$\begin{aligned} \ddot{X}(t) &= F_1(t, X, Y, \dot{X}, \dot{Y}, \mathbf{A}) \\ \ddot{Y}(t) &= F_2(t, X, Y, \dot{X}, \dot{Y}, \mathbf{B}) \end{aligned} \tag{1}$$

where $\mathbf{A}(t)$ and $\mathbf{B}(t)$ are known or assumed Gaussian vector processes which enter as parameters, and $X(0)$, $Y(0)$, $\dot{X}(0)$, $\dot{Y}(0)$ are given Gaussian initial distributions. We want to find a numerical solution of (1) which consists of a description of the joint distribution of the state $(X, Y, \dot{X}, \dot{Y})$ at each time increment t_i, $i = 1,..., N$, given that at t_{i-1} the state was at its mean. In other words, we seek the discrete mean trajectory and covariances. Thus, we call the set $\{\mu_x(t_i), \mu_y(t_i), \mu_{\dot{x}}(t_i), \mu_{\dot{y}}(t_i), \sigma_{xx}(t_i), \sigma_{xy}(t_i),..., \sigma_{\dot{y}\dot{y}}(t_i)\colon i = 1,..., N\}$ a *numerical solution* of (1).

In order to obtain a numerical solution we first convert to a first-order system by putting [2, p. 311]

$$\begin{aligned} U_1(t) &= X(t), \qquad U_2(t) = \dot{X}(t) \qquad U_5(t) = A(t) \\ U_3(t) &= Y(t), \qquad U_4(t) = \dot{Y}(t) \qquad U_6(t) = B(t). \end{aligned} \tag{2}$$

Upon substituting these into (1) we obtain the state space equation for the system (where we have taken the dimension of $\mathbf{A}$ and $\mathbf{B}$ to be 1 without loss of generality).

$$\begin{aligned} \dot{U}_1(t) &= U_2(t) \equiv H_1, & U_1(0) &= X(0) \\ \dot{U}_2(t) &= F_1(t, U_1,..., U_6) \equiv H_2, & U_2(0) &= \dot{X}(0) \\ \dot{U}_3(t) &= U_4(t) \equiv H_3, & U_3(0) &= Y(0) \\ \dot{U}_4(t) &= F_2(t, U_1,..., U_6) \equiv H_4, & U_4(0) &= \dot{Y}(0). \end{aligned} \tag{3}$$

We may express this system by

$$\dot{\mathbf{U}}(t) = \mathbf{H}(t, \mathbf{U}), \qquad \mathbf{U}(0) = \mathbf{U}_0.$$

With $U_j(0)$ Gaussian, $1 \leqslant j \leqslant 4$ and $U_k(t)$ Gaussian, $5 \leqslant k \leqslant 6$, for each t, being known or assumed a priori.

Second, we discretize the system to obtain:

$$0 = t_0 < t_1 < \cdots < t_N = b \qquad \text{(equally spaced)}.$$

We denote $U_j(t_i)$ by U_j^i for $1 \leqslant j \leqslant 6$ and $1 \leqslant i \leqslant N$.

Third, we choose a numerical summer/integrator G to integrate from t_i to t_{i+1}:

$$\begin{aligned} U_1^{i+1} &= G_1(U_1^i, \ldots, U_6^i) \\ &\vdots \qquad\quad \vdots \\ U_4^{i+1} &= G_4(U_1^i, \ldots, U_6^i). \end{aligned} \tag{4}$$

The next step consists of expanding each random integrator $G_j(U_1^i, \ldots, U_6^i)$ at time t_i in a quadratic Taylor polynomial about the mean vector

$$(U_1^i, \ldots, U_6^i) = (\mu_1^i, \ldots, \mu_6^i)$$

where μ_j^i is the mean value of U_j^i. We obtain

$$\begin{aligned} U_j^{i+1} = G_j &+ \sum_{r=1}^{6} (\partial G_j/\partial U_r)(U_r^i - \mu_r^i) \\ &+ \frac{1}{2} \sum_r (\partial^2 G_j/\partial U_r^2)(U_r^i - \mu_r^i)^2 \\ &+ \sum_s \sum_r (\partial^2 G_j/\partial U_r\, \partial U_s)(U_r^i - \mu_r^i)(U_s^i - \mu_s^i). \end{aligned} \tag{5}$$

Fifth, we take the expected value operator on each side of Eq. (5) to obtain

$$\mu_j^{i+1} = G_j + \frac{1}{2} \sum_r (\partial^2 G_j/\partial U_r^2)\, \sigma_{rr}^i + \sum_s \sum_r (\partial^2 G_j/\partial U_r\, \partial U_s)\, \sigma_{rs}^i \tag{6}$$

where σ_{rs}^i is the covariance of U_r and U_s at time t_i.

Step 6 consists of multiplying Eq. (5) by itself and taking the expected value of each side of the result to get

$$\begin{aligned} \sigma_{rs}^{i+1} = &\sum_k (\partial G_r/\partial U_k)(\partial G_s/\partial U_k)\, \sigma_{kk}^i \\ &+ \frac{1}{2} \sum_k (\partial^2 G_r/\partial^2 U_k)(\sigma_{kk}^i)^2 \\ &+ \sum_m \sum_k (\partial G_r/\partial U_k)(\partial G_s/\partial U_m)\, \sigma_{mk}^i - \mu_r^i \mu_s^i. \end{aligned} \tag{7}$$

By means of Eqs. (6) and (7) we can start off at the initial condition $\mathbf{U}_0$ and carry the trajectory state and its accomanying second-order statistics forward in time, stepwise.

3. Error of Approximation

Underlying the random process $X(t)$ is a probability space (S, A, P) where S is the sample space, A is a σ-algebra of events, and P is a real-valued measure on A. $X(t_0)$ is, for fixed t_0, a random variable, i.e., an A-B measurable function from S to the real numbers (B = Borel sets). The mean-square norm of $X(t_0)$ is [13, p. 74] defined via

$$\|X(t_0)\| = E(X^2(t_0))^{1/2} = \int X^2 \, dF \tag{8}$$

where E is the expected value operator and F is the cumulative distribution function for $X(t_0)$. When we have the random vector $\mathbf{X}(t_0) = (X_1(t_0),..., X_m(t_0))$ we use the norm

$$\|\mathbf{X}(t_0)\| = \max_i \{\|X_i(t_0)\|\}. \tag{9}$$

For t_0 in $(0, b)$ the continuity, derivative, and integral of the random process $X(t)$ are defined by the existence of the following limits and values:

$$\lim_{h \to 0} \|X(t_0 + h) - X(t_0)\| = 0 \tag{10}$$

$$\lim_{h \to 0} \|(X(t_0 + h) - X(t_0))/h - D\| = 0 \tag{11}$$

$$\lim_{L \to \infty} \left\| \sum_1^L H(t_i, X(t_i)) \, \Delta t - I \right\| = 0 \tag{12}$$

where D and I are random variables usually designated by $\dot{X}(t_0)$ and $\int_0^{t_0} H(t, X) \, dt$, respectively. We note that the process $X(t)$ is a function from $(0, b)$ to $L_2(P)$. The magnitude of each $X(t)$ is $\|X(t)\|$ and the sup norm of $X(t)$ over $(0, b)$ is

$$\|X\|_\infty = \sup_t \|X(t)\|. \tag{13}$$

The system

$$\dot{X}(t) = F(t, X(t), \mathbf{A}(t)), \qquad X(0) = X_0 \tag{14}$$

is known to have a mean-square solution $X(t)$ in $C((0, b), L^2(P))$ whenever F

satisfies a mean square Lipschitz condition [13, p. 118]. In that case the solution is given by the equivalent equation

$$X(t) = X_0 + \int_0^t F(s, X(s), A(s))\, ds \tag{15}$$

where the integral is the mean-square limiting sum of random variables. We assume here that a mean-square solution $X(t)$ exists for Eq. (14).

THEOREM 1. *Let the system given by Eq.* (14), *or equivalently by* (15), *have Gaussian mean-square solution* $Z(t)$ *on* $T = (0, b)$. *We let* $Y^i = Y(t_i)$ *be the Gaussian sequence determined by our method of solution per Eqs.* (6) *and* (7). *Suppose that the function* $F(t, X(t), A(t))$ *is quadratic in the random variables* $X(t)$ *and* $A(t)$. *Then given any* $\varepsilon > 0$ *there is a discretization* $0, t_1, \ldots, b$ *of* $N(\varepsilon) + 1$ *equally spaced points such that for any* t_0 *fixed in* T *there is a* t_k *in the discretization that satisfies*

$$\| Z(t_0) - Y^k \| < \varepsilon.$$

Proof. We have that

$$\begin{aligned} \| Z(t_0) - Y(t_k) \| &= \| Z(t_0) - Y^k \| \\ &\leqslant \| Z(t_0) - Z(t_k) \| + \| Z(t_k) - G(Y^{k-1}) \| \\ &\quad + \| G(Y^{k-1}) - Y^k \| \end{aligned}$$

where $G(Y^{k-1})$ is the summer/integrator, Y^k is the quadratic expansion of $G(Y^{k-1})$, and $Z(t_k)$ is the solution

$$Z(t_k) = \int_0^{t_k} F(t, Z(t), A(t))\, dt + X_0.$$

By mean-square continuity of the mean-square solution $Z(t)$, we can take a discretization sufficiently fine of equally spaced points $0 = t_1, \ldots, t_m = b$ and a t_k from the discretization such that

$$\| Z(t_0) - Z(t_k) \| < \varepsilon/3.$$

Next we have that

$$\begin{aligned} \| Z(t_k) - G(Y^{k-1}) \| &= \left\| X_0 + \int_0^{t_k} F(t, Z(t), A(t))\, dt \right. \\ &\quad \left. - (X_0 + \sum F(t_i, Z(t_i), A(t_i))\, \Delta t_i) \right\| \\ &= \left\| \int F\, dt - \sum F_i\, \Delta t_i \right\|. \end{aligned}$$

Because the integral exists in the mean-square sense we can refine the above discretization, if needed, by adjoining the midpoints between consecutive points until the last norm value above is less than $\varepsilon/3$.

The final normed difference to consider is

$$\|G(Y^{k-1}) - Y^k\|$$

where Y^k is the random variable obtained by taking the quadratic expansion of the summer/integrator $G(Y^{k-1})$. If $G(Y^{k-1})$ is quadratic in the random variables Y^j and $A(t_j)$ then the value of the last norm is zero. Thus, the result is true.

COROLLARY. *Under the hypotheses of Theorem* 1 *the pointwise mean-square limit of* $Y(t_k)$ *is* $Z(t_0)$, *i.e.*,

$$\lim_{\Delta t \to 0} Y(t_k) = Z(t_0) \qquad \textit{(mean-square)}$$

for each t_0.

4. Computational Aspects and Examples

The partial derivatives in Eqs. (6) and (7) may be computed by the standard 3-point formulas [1, p. 353]. This would require $2M + 1$ functional evaluations for each function G_j at each time step, where M is the number of random variables in the state space. Approximate expressions for the partial derivatives of these integrators may be used, e.g., we may approximate G_j with the Euler formula

$$\tilde{G}_j^{i+1} = U_j^i + H_j(t_i, U^i)\Delta t$$

(see Eq. (3)) and take the analytical expressions for their derivatives. The best way to compute these partial derivatives is to evaluate the analytical expressions for the partials of the G_j.

EXAMPLE 1. Let $\dot{X}(t) = X(t) + U(t)$ on $(0, 1)$, where $X(0)$ is Gaussian with $\mu_0 = 1$ and $\sigma_0 = 0.1$, and $U(t)$ is an independent constant Gaussian process with $\mu_U = 0$ and $\sigma_U = 0.5$. This is a linear system with mean $\mu(t) = e^t$. Table 4.1 shows the results at $t = 1$ for four different cases, as noted in the comments (which tell which random variations were turned off). The derivatives were evaluated numerically.

We note first that the computed $\sigma(1)$ in Case 4 can be checked by taking the root-mean-square (RMS) value of the $\sigma(1)$ values in Case 2 and Case 3, which were run with only one noise source each. Next, we note that $\sigma(1)$ can be found exactly in Case 2 via $E(X(1)^2) = E(X_0^2 \exp(1)^2)$, so that $\sigma^2(1) = E(X_0 \exp(1) - \exp(1))^2) = 0.01 \exp(2) = (0.271828)^2$.

TABLE 4.1

Case	Computed $\mu(1)$	True $\mu(1)$	Computed $\sigma(1)$	True $\sigma(1)$	Comments
1	2.71828	2.71828	0	0	$\sigma(0)=\sigma_U(t)=0$
2	2.71828	2.71828	0.271829	0.271828	$\sigma_U(t)=0$
3	2.71828	2.71828	0.282483	—	$\sigma(0)=0$
4	2.71828	2.71828	0.39203	—	Both noises are on

EXAMPLE 2. Consider the system

$$\dot{X}(t)=0, \qquad X(0)=0$$
$$\ddot{Y}(t)=-g, \qquad Y(0) \text{ is Gaussian}$$
$$\mu_Y(0)=100 \text{ feet}$$
$$\sigma_Y^2(0)=9 \text{ feet.}$$

This system represents the friction-free fall of an object under the influence of gravity ($g=32\,\text{ft/sec}^2$) from a starting position that is Gaussian with center at 100 feet above the ground and with dispersion of $\sigma=3$ feet.

The distance that the object falls in time Δt is a constant value no matter where it started from. After 1 second of fall the object will be at 87, 84, or 83 feet if it started, respectively, from 103, 100, or 97 feet (it will fall 16 feet the first second). Thus, the variance remains fixed.

Table 4.2 shows the computed results. We used the Runge–Kutta integrator of order 4 for G_j, but we obtained the partial derivatives of G_j by taking the analytical partial derivative of the Euler integrator. The results shown in Table 4.2 were computed with a step size $\Delta t=0.5$ on a TRS-80 microcomputer which rounds off to the nearest integer when within 1 on the sixth digit of precision. Because of this rounding fix the computed answers are exact.

TABLE 4.2

t	$\mu_Y(t)$	$\sigma_Y^2(t)$	$\mu_{\dot{Y}(t)}$	$\sigma_{\dot{Y}}(t)$
0	100	9	0	0
0.5	96	9	−16	0
1.0	84	9	−32	0
1.5	64	9	−48	0

EXAMPLE 3. In this nonlinear example, we take

$$\ddot{X}(t) + A(t)(\dot{X}(t)^2 + \dot{Y}(t)) = 0$$
$$\ddot{Y}(t) - B(t)\,\dot{Y}(t) - g = 0$$

where A and B are constant Gaussian processes; $X(0)$, $\dot{X}(0)$, $Y(0)$, $\dot{Y}(0)$ are Gaussian; and $g = 32\ \text{ft/sec}^2$.

This system can be considered as a special type of model of anisotropic projectile motion with a given initial muzzle velocity and no further driving force. The viscous damping parameters are A and B. Table 4.3(A) shows the initial conditions, while Table 4.3(B) shows the computed results. We used the analytic expression for the derivatives of the Euler integrator for convenience. The time step size for the numerical integration is $\Delta t = 0.4$ and the computer was a TRS-80 microcomputer.

From Table 4.3(B) it is unclear whether or not the variances are stabilizing in that they approach limits or continue to grow. Such questions can only be answered analytically. A 95.4% confidence band can be determined by taking $\mu_X \pm 2\sigma_X$ and $\mu_Y \pm 2\sigma_Y$ at each t.

TABLE 4.3(A)

Initial Conditions

0	μ	σ^2
$X(0)$	0	100
$\dot{X}(0)$	2000	400
$Y(0)$	100	9
$\dot{Y}(0)$	0	16
$A(t)$	0.0003	0
$B(t)$	0.032	0.0001

TABLE 4.3(B)

Computed Values

t	μ_X	σ_X^2	μ_Y	σ_Y^2
0	0	100	100	9
0.4	716.89	164	97.429	11.56
0.8	1306.6	181.31	89.672	14.186
1.2	1807.5	187.81	76.662	16.879
1.6	2242.5	190.78	58.332	19.643

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